MATHEMATICAL MODELLING AND THE TRANSMISSION DYNAMIC OF HIV/AIDS WITH EXPOSED AND DEATH COMPARTMENTS

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Abstract: This research presents a Mathematical Modelling and the Transmission Dynamic of HIV/AIDS with Exposed and Death compartments. The model is governed by a system of four ordinary differential equations namely: Susceptible (S), Exposed (E), Infected (I) and Removal or Death (R). An analysis was performed on the model to determine the disease-free equilibrium, endemic equilibrium, reproduction number, local stability, and global stability. The results of the analysis indicate that the disease-free equilibrium equation is locally asymptotically stable if the reproduction number is less than one, and unstable if the reproduction number is greater than one. The system has achieved the condition of being globally asymptotically. We have conducted numerical simulations of the model, and the results show that the susceptible population if they maintain their safe sex life will be removed from the population without being infected with the disease. Exposed at the beginning was very high but due to awareness and the danger of HIV/ AIDS people started to maintain a safe sex life which resulted in a total decline of the disease. Still, they cannot return to their origin because of no cure for the disease. The infected continue moving higher due to unsafe sex life and decline a little because of the safe sex life and precaution. The removal or death gave us almost a straight line because if all the infected people can die the disease will go to extinction.

Keywords: Exposed, awareness, unsafe sex life, Global Stability, Safe sex life.

Introduction

HIV is a viral infection that targets the immune system, specifically the CD4 cells which are white blood cells. This disease damages the CD4 cells, thereby weakening the body's ability to fight off opportunistic infections such as tuberculosis, fungal infections, severe bacterial infections, and certain types of cancers (WHO, 2023; Arias *et.al.*,2022).When the immune system is unable to perform its duty of defending the body against infections and diseases, it is considered deficient. Individuals with immunodeficiency are more susceptible to various infections and cancers that are uncommon in individuals without immunodeficiency.Diseases associated with severe immunodeficiency are opportunistic infections, that take advantage of a weakened immune system.AIDS, or acquired immunodeficiency syndrome, is a condition where the immune system is weakened. It is caused by the human immunodeficiency virus (HIV). The level of immunodeficiency or the appearance of certain infections are used as indicators that HIV infection has progressed to AIDS (WHO, 2023; UNAIDS,2023; Omale *et.al.*, 2019).The United States Centre for Disease Control and Prevention defines AIDS as a condition that occurs when the CD4-positive T-cell count drops below 200 cells per mm3 of blood. These cells are essential in mounting an effective immune response to infections (CDC, 1993).HIV is a virus that can be present in body fluids like blood, semen, vaginal fluids, and breast milk. It spreads through penetrative sex (vaginal or anal), blood transfusion, and sharing of contaminated needles in healthcare settings, and from mother to child during pregnancy, childbirth, and breastfeeding (CDC, 1993; WHO, 2023). Global HIV statistics show that 39 million [33.1 million–45.7 million] people globally were living with HIV in 2022, 1.3 million [1 million–1.7 million] people became newly infected with HIV in 2022 and 630,000 [480 000–880 000] people died from AIDS-related illnesses in 2022 (UNAIDS, 2022) The WHO African Region has the highest number of people living with HIV, with 25.7 million individuals affected in 2018. Additionally, almost two-thirds of new HIV infections worldwide happen in this region. In 2018, approximately 1.1 million people in the African Region were infected with HIV (WHO, 2023). The country with the highest HIV AIDS rate in Africa is Eswatini with a rate of 19.58%, followed closely by Lesotho at 18.72%.Other countries with significant HIV rates include Botswana (15.75%), South Africa (14.75%), and Namibia (8.9%) (HIV rates by the country, 2023; Espitial *et. al.*, 2022) According to estimates, 2.1% (95% CI: 1.5-2.7%) of adults aged 15-49 years in Nigeria are living with HIV. This equates to around 2 million individuals, which is a notable increase from previous estimates of 1.4% based on data from the 2018 NAIIS and UNAIDS projection package. Akwa Ibom has consistently held the top position as the state with the highest HIV burden in Nigeria, with a rate of 5.5%, based on the Nigeria HIV/AIDS Indicator and Impact Survey (Nigeria HIV/AIDS Indicator and Impact Survey, 23; Attaullah *et. al.*, 20220). There are various symptoms of HIV, but not everyone will experience the same ones. The symptoms depend on the individual and the stage of the disease they are in. Two to four weeks after contracting HIV, almost two thirds of individuals may experience a flu-like sickness during stage one, also known as the acute HIV infection. This is how HIV infection naturally affects the body. Flu-like symptoms can include sore throat, muscle pains, fever, chills, rash, night sweats, exhaustion, swollen lymph nodes, and mouth ulcers.In the second stage of clinical latency, the virus continues to

multiply, albeit extremely slowly. At this point, a person may not have any symptoms or feel ill. HIV infection at this stage is also referred to as chronic. People can continue in this stage for ten or fifteen years without HIV therapy, while some people go through this stage more quickly.AIDS in Stage Three If an individual with HIV is not receiving treatment, the virus will eventually erode the immune system and lead to the development of AIDS (acquired immunodeficiency syndrome). The advanced phase of HIV infection is this. AIDS symptoms include sudden weight loss, excessive and inexplicable fatigue, recurrent fever or night sweats, persistent lymph gland enlargement in the neck, groin, or armpits, prolonged diarrhea lasting more than a week, sores on the genitalia, anus, or mouth Asthma, Blotches on the skin, under the skin, inside the mouth, nose, or eyelids that are red, brown, pink, or purplish, Depression, memory loss, and other neurological conditions (Arias *et.al.,* 2022; Kimbir *et. al.,* 2008). Antiretroviral therapy is the most effective treatment for HIV. (ART). This is a concoction of multiple medications designed to regulate the quantity of virus in your body. Antiretroviral drugs reduce the virus's growth rate. While taking these medications can help you stay healthy and lessen the quantity of virus in your body, they cannot treat the illness (Ayele *et. al.*, 2021).

Mathematical Formulation

In this paper we divided the model into four compartments namely: Susceptible S(t), Exposed E(t), Infected I(t) , Removal (death) R(t).It is assumed that there is no recovery once infected, it is assumed that the population is not fixed, and it is assumed that the rate of the disease spread is related to vertical and horizontal interaction, it is assumed that the people in the death compartment are still alive but they cannot survive the disease.The recruitment into the compartment at the rate Λ.The susceptible S(t) increases due to the movement into the compartment through recruitment and exposure at the rates Λ and θ respectively and decreases due to the force of infection and natural death at the rates $\frac{\beta IS}{N}$ and μ. Each compartment suffers natural and induced death rates except Susceptible the susceptible that suffer only natural death rate. The exposure increases due to the progression from susceptible to exposed at the rate $\frac{\beta IS}{N}$ and decreases due to the movement out of the compartment at the rates $(\theta+\gamma+\mu+\delta)$. Where θ is the progression back to the susceptible compartment, γ is the progression from exposed to infected compartment, μ is the natural death rate and δ is induced death rate. The infected compartment increases due to the progression from exposure to the infected compartment at the rate γ and decreases at the rate $(\alpha+\mu+\delta)$. where α is the progression to the Removal compartment. The death compartment increased due to the progression into the compartment at the rate α and decreased at the rates $(\mu+\delta)$.

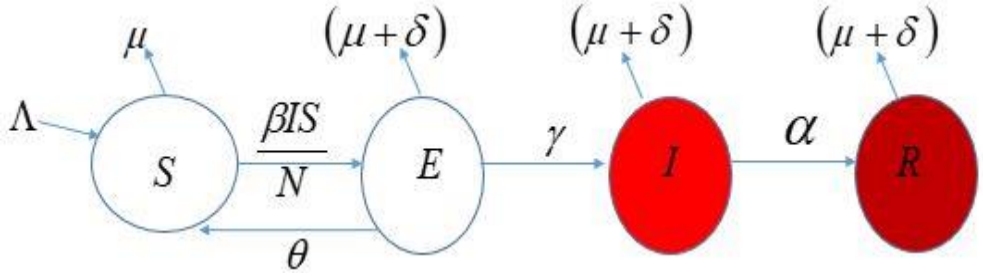

Figure 2 Schematic Diagram of the Model

Equations of the Model

$$\left.\begin{aligned}\frac{dS}{dt}&=\Lambda+\theta E-\mu S-\frac{\beta SI}{N}\\ \frac{dE}{dt}&=\frac{\beta SI}{N}-A_1E\\ \frac{dI}{dt}&=\gamma E-A_2I\\ \frac{dR}{dt}&=I\gamma-A_3R\end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} A_1 &= (\mu+\delta+\gamma+\theta) \\ A_2 &= (\mu+\delta+\gamma) \\ A_3 &= (\mu+\delta) \end{aligned}\right\} \quad (2)$$

With initial conditions;

$$S(0)=S_0, E(0)=E_0, I(0)=I_0, R(0)=R_0.$$

Table:1 Variables and parameters of the model

Par.	Description	Values	Ref.
β	Contact rate	0.0000003	Arias *et.al.*,2022
δ	Induced	0.000005	Arias *et.al.*,2022
μ	Natural death rate	0.02	Ibrahim *et. al.,* 2021
N	Total population	2 138 500	Ayele *et. al.*, 2021
γ	Progression from exposure to infected	0.5	Ibrahim *et. al.*, 2021
θ	Progression from exposed back to the susceptible	0.03	Assumed
α	Progression from infected to death	0.000012	Arias *et.al.*,2022
Λ	Recruitment	0.55	Assumed

Model Analysis

In this section, we obtain the following results which guarantee that the HIV/AIDS diseases is governed by the system (1) is epidemiologically and mathematically well-posed in a feasible region given by:

$$S_0 = \left\{ \{S,E,I,R\} \in R_+^4 \; S>0, E\geq 0, I\geq 0, R\geq 0, N_{(t)} \leq \frac{\Lambda}{\mu} + \left(\mu_0 - \frac{\Lambda}{N}\right)e^{\mu t} \right\}$$

Disease Free and Endemic Equilibrium Points of the HIV/AIDS Model

We find the equilibrium points by setting the right-hand side of the system (1.0) to zero as follows:

$$\left.\begin{aligned} \Lambda+\theta E-\mu S-\frac{\beta SI}{N}&=0 \\ \frac{\beta SI}{N}-A_1E&=0 \\ \gamma E-A_2I&=0 \\ I\gamma-A_3R&=0 \end{aligned}\right\} \quad (3)$$

When there is no disease in the population $(E(t)=I(t)=R(t)=0)$, the disease-free equilibrium (DFE) point of the HIV/AIDS model (1.0) exist and is given by $E_0=(S_0,E_0,I_0,R_0)=\left\{\frac{\Lambda}{\mu},0,0,0\right\}$

To compute the endemic equilibrium point (EEP), from equation (1) and (2) we have:

$$\left.\begin{aligned} S &= \frac{\Lambda A_1A_2}{\beta\gamma\mu} \\ E &= \frac{(\Lambda\beta\gamma\mu+\theta\lambda A_1A_2-\lambda\mu A_1A_2)}{\beta\gamma\mu A_1} \\ I &= \frac{(\Lambda\beta\gamma\mu+\theta\Lambda A_1A_2-\Lambda\mu A_1A_2)}{\beta\mu A_1A_2} \\ R &= \frac{(\theta\alpha\Lambda A_1A_2-\alpha\Lambda\mu A_1A_2+\alpha\Lambda\beta\gamma\mu)}{\beta\mu A_1A_2A_3} \end{aligned}\right\} \quad (4)$$

Basic Reproduction Number

The basic reproduction number denoted by R_0 is the expected number of secondary cases produced in a complete susceptible population by a typical infective individual.

Let $F(x)$ be the rate of appearance of new infections in compartment $V_I^*,(x)$ be the rate of transfer of individuals out of the compartment.

It can be calculated as $R_0=\eta(FV^{-1})$where

$$F = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$ and

$$V = \begin{pmatrix} (\theta+\mu+\delta+\gamma) & 0 \\ -\gamma & (\mu+\delta+\gamma) \end{pmatrix}$$

and

$$FV^{-1} = \begin{pmatrix} \dfrac{\beta\gamma}{(\gamma+\mu+\delta)(\theta+\gamma+\mu+\delta)} & \dfrac{\beta}{\gamma+\mu+\delta} \\ 0 & 0 \end{pmatrix}$$

Hence the reproduction number for the Syphilis is the dominant Eigen value, thus $R_{0M} = \eta(FV^{-1})$, where ρ is the spectral radius such that

$$\therefore R_0 = \frac{\beta\gamma}{(\gamma+\mu+\delta)(\theta+\gamma+\mu+\delta)} \qquad (5)$$

Local Stability of the Disease-Free Equilibrium of the HIV/AIDS Model

Proposition 1: The disease-free equilibrium E_0 point of HIV/AIDS model is locally asymptotically stable if $R_0 < 1$ and unstable if$R_0 > 1$. This proposition is achieved by$|J(E_0) - \lambda I| = 0$. From (1) we have

$$\begin{pmatrix} -\mu & \theta & -\beta & 0 \\ 0 & -A_1 & \beta & 0 \\ 0 & \gamma & -A_2 & 0 \\ 0 & 0 & \alpha & -A_3 \end{pmatrix} \qquad (6)$$

$$\begin{pmatrix} \mu-\lambda_1 & 0 & -\beta & 0 \\ 0 & -A_1-\lambda_2 & \beta & 0 \\ 0 & \gamma & -A_2-\lambda_3 & 0 \\ 0 & 0 & \alpha & -A_3-\lambda_3 \end{pmatrix} \qquad (7)$$

Using reduction method, we have

$$\begin{pmatrix} -A_1-\lambda_2 & \beta \\ \gamma & -A_2-\lambda_3 \end{pmatrix} \qquad (8)$$

$$\lambda_1 = -\mu,\ \lambda_4 = -A_3$$

We applied Routh-Hurwitz criterion which stated that all roots of the polynomial (8) have negative real parts if and only if the coefficient, A_i are positive and the determinant of the matrices $H_i > 0$ for $i = 1,2,...,n_i$

$$\begin{pmatrix} -A_1 & \beta \\ \gamma & -A_2 \end{pmatrix} \qquad (9)$$

Characteristic polynomial of (9), we have

$$B_1X^2 + B_2X + B_3$$

$$B_1 = 1$$

$$B_2 = (A_1 + A_2)$$

$$B_3 = (A_1A_2 - \beta\gamma)$$

$$H = \begin{vmatrix} B_1 & B_3 & B_5 \\ 1 & B_2 & B_4 \\ 0 & B_1 & B_3 \end{vmatrix} \qquad (10)$$

$$H_1 = B_1 > 0$$

$$H_2 = \begin{pmatrix} B_1 & B_3 \\ 1 & B_2 \end{pmatrix}, B_1B_2 > B_3$$

$$H_2 > 0$$

$$H_3 = \begin{pmatrix} B_1 & B_3 & 0 \\ 1 & B_2 & 0 \\ 0 & B_1 & B_3 \end{pmatrix}, B_1B_2B_3 > B\frac{2}{3}, \quad H_3 > 0$$

Since $H_1 = H_2 = H_3 > 0$, hence the HIV/AIDS is locally asymptomatically stable.

Global Stability of the disease-free equilibrium point

For global stability of the DFE, the technique used by Castilo-Chavez *et al.,*(2002). was employed. The model is rewritten as follows:

$$\left.\begin{aligned} \tfrac{dX}{dt} &= K(X,Z) \\ \tfrac{dZ}{dt} &= G(X,Z),\ G(X,0) = 0 \end{aligned}\right\} \qquad (12)$$

where $X \in \mathbb{R}^2$ and $X = \{S,R\}$ denotes the number of uninfected individuals and $Z \in \mathbb{R}^2$ where $Z = \{E,I\}$ denotes the number of infected individuals. $E^0 = \left\{\frac{\Lambda}{\mu}, 0,0,0\right\}$ denotes the disease-free equilibrium point of this system, where,

$$x^* = \left\{\frac{\Lambda}{\mu}\right\} \qquad (13)$$

Condition (10) may be met to guarantee global asymptotic stability

(H_1): For $\frac{dx}{dt} = k(x,0), x^*$is globally asymptotic stable.

(H_2): For $G(X,Z) = AZ - \hat{G}(X,Z), \hat{G}(X,Z) \geq 0 \forall (x,z) \in \Gamma$ where $A = D^G(x^*,0)$ is an m matrix and Γ is the region where the model has biological meaning.

Theorem 1. If the system (1) satisfies condition (12), then the fixed point $E^0 = \left\{\frac{\Lambda}{\mu}, 0,0,0,0\right\}$is a globally

asymptotically stable equilibrium of the system (1) provided that $R_o < 1$ and the conditions (H_1) and (H_2) are satisfied:

Proof

Consider $K(X,0) = [\Lambda - \mu S]$ and $G(X,Z)$ and $G(X,Z) = AZ - \hat{G}(X,Z)$

$$A = \begin{pmatrix} -(\theta+\mu+\delta+\gamma) & \frac{\beta S}{N} & 0 \\ \gamma & -(\mu+\delta+\gamma) & 0 \\ 0 & \alpha & -(\mu+\delta) \end{pmatrix} \quad (14)$$

and

$$Z = \begin{bmatrix} E \\ I \\ R \end{bmatrix} \quad (15)$$

$$G(X,Z) = \begin{vmatrix} \frac{BSI}{N} & -(\theta+\mu+\delta+\gamma)E \\ \gamma E & -(\mu+\delta+\alpha)I \\ \alpha I & -(\mu+\alpha)R \end{vmatrix} \quad (16)$$

Given

$$\hat{G}(X,Z) = AZ - G(X,Z) \quad (17)$$

Substituting equations (12), (13), and (14) into (15) we have

$$\hat{G}(X,Z) = \begin{bmatrix} \frac{BSI}{N} & -(\theta+\mu+\delta+\gamma)E \\ \gamma E & -(\mu+\delta+\alpha)I \\ \alpha I & -(\mu+\alpha)R \end{bmatrix} - \begin{bmatrix} \frac{BSI}{N} & -(\theta+\mu+\delta+\gamma)E \\ \gamma E & -(\mu+\delta+\alpha)I \\ \alpha I & -(\mu+\alpha)R \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (18)$$

Since all the conditions are satisfied, $\hat{G}(X,Z) \geq 0$, the DFE, E^0 is globally asymptotically stable.

Numerical Simulation

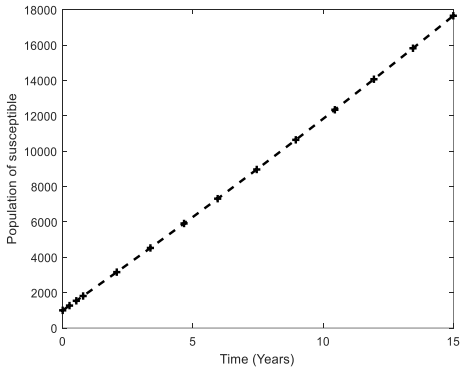

Figure 2 Diagram showing the dynamics of susceptible population

Figure 2 shows almost a straight line but it is not since our equations are nonlinear which indicates that at this time there is no HIV/aids in the population and people die naturally not as result of the diseases.

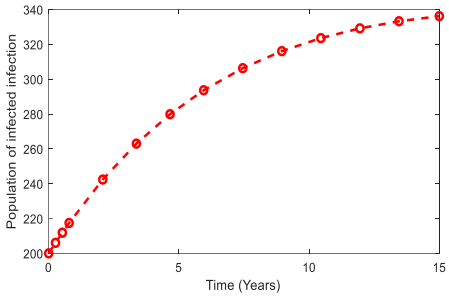

Figure 4 Diagram showing the dynamics of the infected population.

Figure 3 shows that the increase in HIV/aids will continue for up to 15 years if precaution is not put in place.

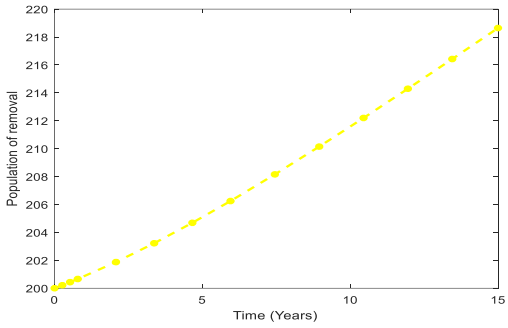

Figure 5 Diagram showing the dynamics of the death population

Figure 5 is the removal or death compartment it tells us that since HIV/ AIDS has no cure all the people who are infected will die either of the disease or naturally.

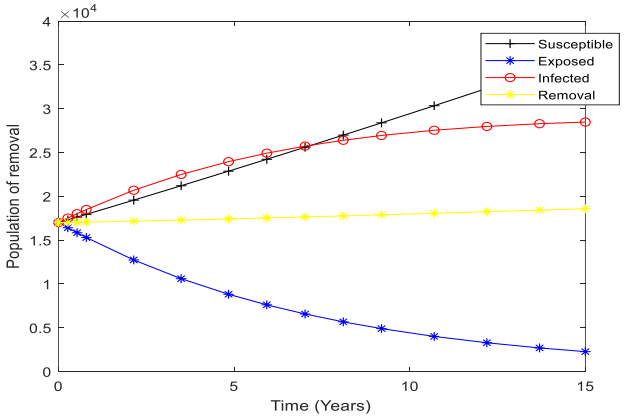

Figure 5 shows the combination of the four compartments, it shows that the susceptible population if they maintain their safe sex life will be removed from the population without being infected with the disease. Exposed at the beginning was very high but due to awareness and the danger of HIV/ aids people started to maintain a safe sex life which resulted in a total decline of the disease but they cannot return to their origin because of no cure for the disease. The infected continue moving higher due to unsafe sex life and decline a little because of the safe sex life and precaution. The removal gave us almost a straight line because if all the infected people can die the disease will go to extinction.

Conclusion

The model was analysed to find the global stability, reproduction number, endemic equilibrium, disease-free equilibrium, and local stability. The analysis's findings show that global stability has been properly attained and that the disease-free equilibrium equation is locally asymptotically stable. Our focus is on safe sexual behaviours within the population, based on numerical simulations of the model. Our graphs' results lead us to emphasize and caution the public about maintaining safe sexual practices and taking appropriate precautions when dealing with HIV/AIDS-related concerns, as this is the only way the illness can quickly go extinct. Our findings suggest that within the next fifteen years,If prevention measures and safe sexual behaviour are put in place, HIV/AIDS will nearly completely disappear.

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